ENERGY TRANSFER FROM A RADIATING SPHERE INTO A MEDIUM WITH MOLECULAR HEAT CONDUCTION

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O. V. VOINOV, A. M. GOLOVIN and A. G. PETROV
(Moscow)
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We consider a problem concerning the temperature existing around a radiating sphere situated in an infinite, homogeneous medium in which, in addition to absorption of radiation, we have molecular energy transfer. Solution is obtained under the condition that the radius a of the sphere is small compared with the path length $1/\alpha$ of the photons within the medium. At the distances from the sphere which are large compared with $1/\alpha$ the solution corresponds approximately to the radiant heat transfer. When molecular transfer is absent in the problems on radiation, then the temperature discontinuity occurs at the surface of the sphere, but this is smoothed out when a finite value is assigned to the molecular heat transfer coefficient. If this coefficient is small compared with the radiant heat transfer coefficient, but their ratio exceeds considerably the value of the parameter αa, then the temperature field near the sphere is defined solely in the terms of the molecular transfer. In this case the energy flux is dominated by the molecular heat transfer near the sphere and by the radiant transfer away from the sphere.

1. Basic equations. A sphere of radius a is assumed to be a grey radiation source whose effective black body coefficient is ε, surrounded by a radiation absorbing gas whose absorption coefficient is α and the temperature away from the sphere is T_∞. Coefficient α represents the radiation absorption coefficient averaged over the whole spectrum and is assumed to be constant.

Radiation intensity $I(r, \theta)$ is obtained from the kinetic equation which, for the spherically symmetric case and in the presence of a local thermodynamic equilibrium, has the form [1]

$$\cos\theta\frac{\partial I}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I}{\partial\theta} = \alpha(I_p - I) \tag{1.1}$$

where $I_p = (\sigma/\pi)T^4$ is the equilibrium radiation intensity, σ is the Stefan-Boltzmann constant, r is the distance between the observer and the center of the sphere and θ is the angle between the direction of the moving photons and the radius vector r.

Let us assume that the intensity of radiation $I_a(\theta)$ emitted by the surface of the sphere is given, and that it includes the radiation emitted by the surface of the grey sphere of temperature T_a as well as the intensity of reflected radiation, the reflection coefficient being equal to $1-\varepsilon$

$$I_a(\theta) = \varepsilon(\sigma/\pi)T_a^4 + (1-\varepsilon)I(a, \pi-\theta) \tag{1.2}$$

At the distance from the sphere the temperature tends to its limiting value T_∞ and the radiation intensity, to the thermodynamic equilibrium radiation intensity

$$I \to (\sigma/\pi)T_\infty^4 \qquad \text{as } r\to\infty \tag{1.3}$$

It was shown in [2] that the solution of (1.1) with the boundary conditions (1.2) and (1.3) has the form

when $0 < \theta < \psi$ $(\psi = \arcsin a/r)$

$$I = I_a(\theta)\exp[-\alpha(r\cos\theta - \sqrt{a^2 - r^2\sin^2\theta})] +$$

$$+\frac{\alpha\sigma}{\pi}\int_a^r \exp\left[-\alpha\left(r\cos\theta-\sqrt{\rho^2-r^2\sin^2\theta}\right)\right]\frac{\rho T^4(\rho)\,d\rho}{\sqrt{\rho^2-r^2\sin^2\theta}}$$

when $1/2\ \pi < \theta < \pi$

$$I=\frac{\alpha\sigma}{\pi}\int_r^\infty \exp\left[-\alpha\left(r\cos\theta+\sqrt{\rho^2-r^2\sin^2\theta}\right)\right]\frac{\rho T^4(\rho)\,d\rho}{\sqrt{\rho^2-r^2\sin^2\theta}}$$

$$I_c=I\left(r\sin\theta,\ \frac{\pi}{2}\right)=\frac{\alpha\sigma}{\pi}\int_{r\sin\theta}^\infty \exp\left(-\alpha\sqrt{\rho^2-r^2\sin^2\theta}\right)\frac{\rho T^4(\rho)\,d\rho}{\sqrt{\rho^2-r^2\sin^2\theta}}$$

when $\psi<\theta<1/2\,\pi$

$$I=I_c\exp\left(-\alpha r\cos\theta\right)+\frac{\alpha\sigma}{\pi}\times$$

$$\times\int_{r\sin\theta}^{r} \exp\left[-\alpha\left(r\cos\theta-\sqrt{\rho^2-r^2\sin^2\theta}\right)\right]\frac{\rho T^4(\rho)\,d\rho}{\sqrt{\rho^2-r^2\sin^2\theta}} \tag{1.4}$$

Radiant energy density U and the radiant energy flux density S in the radial direction are given, by virtue of the spherical symmetry of the problem, by

$$U=\frac{2\pi}{c}\int_0^\pi I\sin\theta\,d\theta,\qquad S=2\pi\int_0^\pi I\cos\theta\sin\theta\,d\theta \tag{1.5}$$

As we know [1 and 3], the equilibrium radiant energy density U_p for the medium at the temperature T and in the state of thermodynamic equilibrium is equal to $4\ (\sigma / c)T^4$.

Integrating the radiation transfer equation (1.1) over the solid angle, we obtain the continuity equation

$$\left(\frac{d}{dr}+\frac{2}{r}\right)S=\alpha c\left(U_p-U\right) \tag{1.6}$$

If molecular heat transfer takes place within the medium in addition to the radiant tranfer, then the law of conservation of energy implies that

$$\left(\frac{d}{dr}+\frac{2}{r}\right)\left(S-\varkappa\frac{dT}{dr}\right)=0 \tag{1.7}$$

where $\varkappa$ denotes the molecular heat transfer coefficient.

Relations (1.6) and (1.7) yield the following equation defining the temperature distribution in the medium

$$\varkappa\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)T=\alpha c\left(U_p-U\right) \tag{1.8}$$

2. Energy and radiant flux densities. Expression (1.5) for the enregy density can be written as

$$U=(2\pi / c)\ (U_1+U_2+U_3)$$

$$\left(U_1=\int_0^\psi I\sin\theta\,d\theta,\quad U_2=\int_{1/2\pi}^{\pi} I\sin\theta\,d\theta,\quad U_3=\int_\psi^{1/2\pi} I\sin\theta\,d\theta\right) \tag{2.1}$$

where the respective integrals are computed with the aid of Formulas (1.4).

Following the example of [2], we utilize the following substitution:

$$r\cos\theta-\sqrt{\rho^2-r^2\sin^2\theta}=u,\qquad r\cos\theta+\sqrt{\rho^2-r^2\sin^2\theta}=v \tag{2.2}$$

$$\frac{\sin\theta\,d\theta}{\sqrt{\rho^2-r^2\sin^2\theta}}=\frac{du}{ru}=-\frac{dv}{rv},\qquad \cos\theta=\frac{r^2+u^2-\rho^2}{2ru}=\frac{r^2+v^2-\rho^2}{2rv}$$

The integrals in (2.1) are simplified to:

$$U_1 = \frac{\varepsilon\sigma}{2\pi r} T_a^4 \int_{r-a}^{q} \left(\frac{r^2 - a^2}{u^2} - 1\right) e^{-\alpha u}\, du +$$

$$+ (1-\varepsilon)\frac{\alpha\sigma}{\pi} \int_a^\infty \rho T^4(\rho) \int_0^\psi E(r, \rho, \theta) \frac{\sin\theta\, d\theta\, d\rho}{\sqrt{\rho^2 - r^2\sin^2\theta}} + \frac{\alpha\sigma}{\pi r} \int_a^r \rho T^4(\rho) \int_{r-\rho}^{q-p} e^{-\alpha u} \frac{du}{u}\, d\rho \quad (2.3)$$

$$(E(r, \rho, \theta) = \exp[-\alpha(r\cos\theta - 2\sqrt{a^2 - r^2\sin^2\theta} + \sqrt{\rho^2 - r^2\sin^2\theta})].$$

$$U_2 = -\frac{\alpha\sigma}{\pi r} \int_r^\infty \rho T^4(\rho) \int_t^{\rho - r} e^{-\alpha v} \frac{dv}{v}\, d\rho \quad (2.4)$$

$$U_3 = -\frac{\alpha\sigma}{\pi r} \int_a^r \rho T^4(\rho) \int_{q+p}^{t} e^{-\alpha v} \frac{dv}{v}\, d\rho +$$

$$+ \frac{\alpha\sigma}{\pi r} \int_a^r \rho T^4(\rho) \int_{q-p}^{t} e^{-\alpha u} \frac{du}{u}\, d\rho - \frac{\alpha\sigma}{\pi r} \int_r^\infty \rho T^4(\rho) \int_{q+p}^{t} e^{-\alpha v} \frac{dv}{v}\, d\rho \quad (2.5)$$

$$(q = \sqrt{r^2 - a^2}, \quad p = \sqrt{\rho^2 - a^2}, \quad t = \sqrt{|r^2 - \rho^2|})$$

The expression for the radiant energy density has the form

$$U = \frac{\varepsilon\sigma}{cr} T_a^4 \int_{r-a}^{q} \left(\frac{r^2 - a^2}{u^2} - 1\right) e^{-\alpha u}\, du + \quad (2.6)$$

$$+ 2(1-\varepsilon)\frac{\alpha\sigma}{c} \int_a^\infty \rho T^4(\rho) \int_0^\psi E(r, \rho, \theta) \frac{\sin\theta\, d\theta\, d\rho}{\sqrt{\rho^2 - r^2\sin^2\theta}} + \frac{2\alpha\sigma}{cr} \int_a^\infty \rho T^4(\rho) \int_{|r-\rho|}^{p+q} e^{-\alpha u} \frac{du}{u}\, d\rho$$

The above expression unlike the one given in [2], takes into account the radiation reflected from the surface of the sphere. Both, however, become equivalent for the perfectly black sphere ($\varepsilon = 1$). Similar calculations performed for the radial component of the flux density lead to the following results:

$$S = \frac{\varepsilon\sigma}{2r^2} T_a^4 \int_{r-a}^{q} \left[\left(\frac{r^2 - a^2}{u}\right)^2 - u^2\right] e^{-\alpha u} \frac{du}{u} +$$

$$+ 2(1-\varepsilon)\alpha\sigma \int_a^\infty \rho T^4(\rho) \int_0^\psi E(r, \rho, \theta) \frac{\sin\theta\cos\theta\, d\theta\, d\rho}{\sqrt{\rho^2 - r^2\sin^2\theta}} +$$

$$+ \frac{\alpha\sigma}{r^2} \int_a^\infty \rho T^4(\rho) \int_{|r-\rho|}^{p+q} \left(1 + \frac{r^2 - \rho^2}{u^2}\right) e^{-\alpha u}\, du\, d\rho \quad (2.7)$$

Since $\alpha a \ll 1$ by definition, we can replace, in (2.6) and (2.7), $e^{-\alpha u}$ with $e^{-\alpha r}$ in the first integrals and put $\exp(-2\alpha\sqrt{a^2 - r^2\sin^2\theta})$ equal to unity. Moreover, we can lump the terms independent of ε together and, again using the condition $\alpha a \ll 1$, replace $e^{-\alpha u}$ with $e^{-\alpha(r+\rho)}$ in the second integrals, thus obtain

$$U = \frac{2\varepsilon\sigma}{c} e^{-\alpha r} \left[T_a^4 \left(1 - \sqrt{1 - \frac{a^2}{r^2}}\right) - \frac{\alpha}{r} \int_a^\infty \rho T^4(\rho)\, e^{-\alpha\rho} \ln\frac{r+\rho}{p+q}\, d\rho\right] +$$

$$+\frac{2\alpha\sigma}{cr}\int_a^\infty \rho T^4(\rho)\int_{|r-\rho|}^{r+\rho} e^{-\alpha u}\frac{du}{u}\,d\rho \tag{2.8}$$

$$S=\frac{\varepsilon\sigma a^2}{r^2}e^{-\alpha r}\left[T_a{}^4-\frac{2\alpha}{a^2}\int_a^\infty \rho T^4(\rho)\,e^{-\alpha\rho}\,(\rho-\sqrt{\rho^2-a^2})\,d\rho\right]+$$

$$+\frac{\alpha\sigma}{r^2}\int_a^\infty \rho T^4(\rho)\int_{|r-\rho|}^{r+\rho}\left(1+\frac{r^2-\rho^2}{u^2}\right)e^{-\alpha u}\,du\,d\rho \tag{2.9}$$

The second integral in (2.8) simplifies considerably, if the region $r \gg a$ is considered, since in this case it is the quantities $\rho \sim 1/\alpha$ that make the main contribution towards the integral. Consequently, the expression within the square brackets in (2.8) can be written as

$$\frac{a^2}{2r^2}\left(T_a{}^4-\alpha\int_a^\infty T^4(\rho)\,e^{-\alpha\rho}\,d\rho\right)$$

We note that at the point $r = a$ this expression becomes

$$T_a{}^4-\alpha\int_0^\infty T^4(\rho)\,e^{-\alpha\rho}\,d\rho$$

and this makes it possible to write it as

$$\left[T_a{}^4-\alpha\int_0^\infty T^4(\rho)\,e^{-\alpha\rho}\,d\rho\right](1-\sqrt{1-a^2/r^2})$$

This substitution is valid when $r \gg a$ and when $r = a$. It can be assumed that the deviations from this formula will not be significant in the intermediate region $r \sim a$.

Thus we obtain the following expression for the radial energy density:

$$U=\frac{2\varepsilon\sigma}{c}e^{-\alpha r}\left[T_a{}^4-\alpha\int_a^\infty T^4(\rho)\,e^{-\alpha\rho}d\rho\right]\left(1-\sqrt{1-\frac{a^2}{r^2}}\right)+$$

$$+\frac{2\alpha\sigma}{cr}\int_a^\infty \rho T^4(\rho)\int_{|r-\rho|}^{r+\rho} e^{-\alpha u}\frac{du}{u}\,d\rho \tag{2.10}$$

Analogous treatment applied to the expression for the radiant energy flux density results in

$$S=\frac{\varepsilon\sigma a^2}{r^2}e^{-\alpha r}\left[T_a{}^4-\alpha\int_a^\infty T^4(\rho)\,e^{-\alpha\rho}d\rho\right]+$$

$$+\frac{\alpha\sigma}{r^2}\int_a^\infty \rho T^4(\rho)\int_{|r-\rho|}^{r+\rho}\left(1+\frac{r^2-\rho^2}{u^2}\right)e^{-\alpha u}du\,d\rho \tag{2.11}$$

In the region $\alpha r \gg 1$ the temperature $T(\rho)$ will be a slowly varying function of coordinates, then computing the flux and energy densities we can limit ourselves to the first terms of the Taylor's expansion for $T^4(\rho)$

$$T^4(\rho)=T^4(r)+(\rho-r)(d/dr)\,T^4(r)$$

and taking into account that the integrand function is different from zero only within the region $\alpha\,|r-\rho\,| \lesssim 1$, the lower limit of integration in ρ (for $\alpha r \gg 1$) can be replaced by $-\infty$.

In consequence, the radiant energy density and the radiant energy flux density become

$$U = \frac{4\sigma}{c} T^4(r) + \frac{8\sigma}{3c\alpha^2 r} \frac{d}{dr} T^4(r), \qquad S = -\frac{4\sigma}{3\alpha} \frac{d}{dr} T^4(r) \tag{2.12}$$

from which we see that as $r \to \infty$, the radiant energy density tends to the thermodynamically stable radiant energy density, while the radiant flux density tends to the well known expression for the flux in the approximation to the radiant energy transfer [3].

If the total energy flux $4\pi a^2 S_a$ through the surface of the sphere of radius a is given, then, according to (2.12), Eq. (1.12) in the region $\alpha r \gg 1$ becomes

$$-\left(\varkappa + \frac{16\sigma}{3\alpha} T_\infty{}^3\right) \frac{dT}{dr} = \frac{a^2}{r^2} S_a \tag{2.13}$$

and its solution

$$T - T_\infty = \frac{3\alpha a^2 S_a}{16\sigma T_\infty{}^3 r\,(1 + 3\mu^2\alpha^2)} \qquad \left(\mu^2 = \frac{\varkappa}{16\alpha\sigma T_\infty{}^3}\right) \tag{2.14}$$

will be the principal asymptotic part of the solution for the exact Eq. (1.8).

3. Temperature field at the distance from the surface of the sphere. Let us introduce a new function φ

$$T^4 = T_\infty{}^4 (1 + \varphi), \quad T_a{}^4 = T_\infty{}^4 (1 + \varphi_a) \tag{3.1}$$

Assuming $\varphi_a \ll 1$, we can linearize (1.8) obtaining

$$\mu^2 \frac{d^2}{dr^2} r\varphi = r(1+\varphi) - A\left(r - \sqrt{r^2 - a^2}\right) e^{-\alpha r} -$$

$$- \frac{\alpha}{2} \int_a^\infty \rho (1 + \varphi)\, d\rho \int_{|r-\rho|}^{r+\rho} e^{-\alpha u} \frac{du}{u} \qquad \left(A = \frac{\varepsilon}{2}\left(\varphi_a - \alpha \int_a^\infty \varphi e^{-\alpha\rho}\, d\rho\right)\right) \tag{3.2}$$

The integral appearing in (3.2) can be transformed as follows:

$$\int_a^\infty \rho d\rho \int_{|r-\rho|}^{r+\rho} e^{-\alpha u} \frac{du}{u} = \int_{-\infty}^\infty \rho d\rho \int_{|r-\rho|}^\infty e^{-\alpha u} \frac{du}{u} - \int_0^a \rho d\rho \int_{|r-\rho|}^{r+\rho} e^{-\alpha u} \frac{du}{u} = \frac{2r}{\alpha}\left(1 - \frac{\alpha a^3}{6r^2} e^{-\alpha r}\right)$$

From this we see that, when the integration over ρ in (3.2) is extended to the region $0 < \rho < a$ terms of the order of αa appear. These terms can, however, be neglected, since the initial Eq. (3.2) was obtained by neglecting the terms of the order of αa. This is even more obvious from the fact that a more exact equation analogous to (3.2) should have a solution $\varphi \equiv 0$. Eq. (3.2) can formally be extended over the whole region $0 < \rho < \infty$ and $\varphi(r)$ defined additionally for $r < 0$ by putting $\varphi(r) = \varphi(-r)$. Then the following equation can be written in the region $-\infty < r < \infty$:

$$\mu^2 \frac{d^2 r\varphi}{dr^2} = r\varphi - Ar\left(1 - \sqrt{1 - \frac{a^2}{r^2}}\right) e^{-\alpha|r|} \eta(r^2 - a^2) -$$

$$- \frac{\alpha}{2} \int_{-\infty}^\infty \rho\varphi(\rho) E_1(\alpha|r - \rho|)\, d\rho \tag{3.3}$$

$$\left(E_n(x) = \int_1^\infty e^{-xu} \frac{du}{u^n}, \quad \eta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}\right)$$

We shall use the Fourier transformation

$$\Phi(k) = \int_{-\infty}^\infty r\varphi(r)\, e^{-ikr}\, dr \tag{3.4}$$

to solve (3.3). Let $r\varphi = aB$ as $r \to +0$, then the Fourier transform of (3.3) has the form

$$-\mu^2k^2\Phi - 2\,i\mu^2kaB = \Phi - AF - (\alpha/k)\Phi \operatorname{arc\,tg}(k/\alpha) \qquad (3.5)$$

$$F = \int_a^\infty (r - \sqrt{r^2 - a^2})\, e^{-\alpha r}(e^{-ikr} - e^{ikr})\,dr =$$

$$= 2i \operatorname{Im}\left\{\left[\frac{1}{(\alpha + ik)^2} + \frac{a}{\alpha + ik}\right] e^{-(\alpha+ik)a} - \frac{a}{\alpha + ik} K_1'(\alpha a + ika)\right\} \qquad (3.6)$$

where $K_1(x)$ is the Bessel function of an imaginary argument (i.e. the MacDonald function). If $|k|a \ll 1$, then

$$F(k) = -ia^2 \operatorname{arc\,tg}(k/\alpha) \qquad (3.7)$$

This Fourier transform corresponds formally to the function $\frac{1}{2}(a^2/r)\exp(-\alpha r)$ which is the asymptotic form of the free term of (3.3).

Inverse Fourier transformation yields

$$r\varphi(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(AF - 2i\mu^2kaB)\exp(ikr)\,dk}{1 + \mu^2k^2 - (\alpha/k)\operatorname{arctg}(k/\alpha)} \qquad (3.8)$$

Analysis shows that the integrand function has, on the complex plane, one first order pole $k = 0$ and two branch points $k = \pm i\alpha$. In addition, this function may have poles in the region $|\operatorname{Im} k| > \alpha$ which are roots of Eq.

$$1 + \mu^2k^2 = \frac{\alpha}{2ik}\ln\frac{i\alpha - k}{i\alpha + k} \qquad (3.9)$$

If $\mu\alpha \ll 1$, then (3.9) transforms on the upper semiplane, with the accuracy of up to the terms of the order of $\mu^2\alpha^2$, into an equation whose obvious solution is

$$1 + \mu^2k^2 \mp \tfrac{1}{2}\alpha\pi / k = 0, \qquad k = i/\mu \mp \tfrac{1}{4}\alpha\pi \qquad (3.10)$$

where the upper sign corresponds to going around the branch point anticlockwise and the lower sign, to the clockwise movement. We see from (3.10) that these roots do not lie on the principal branch of the logarithmic curve. Singularities of (3.9) are distributed on the k-plane symmetrically with respect to the real axis.

Methods of conformal transformations applied to the functions appearing on both sides of (3.9) show, that for any value of $\mu\alpha$ the only singulatities of (3.9) are: one second order pole at the origin and two branch points $k = \pm i\alpha$.

The path of integration passes through the pole at $k = 0$. Integral (3.8) should be taken as a half-sum of the integral along the real axis with an indentation above and below the pole at $k = 0$. The residue of this pole defines the behvior of the function as $|r| \to \infty$.

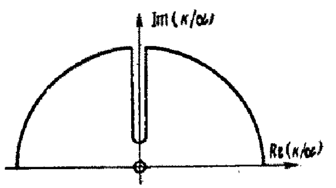

Fig. 1

Parity of the function $\varphi(r)$ implies that the following condition holds:

$$\lim_{r\to\infty} r\varphi(r) = -\lim_{r\to-\infty} r\varphi(r)$$

although it will obviously not hold, when any other path of integration is chosen in the vicinity of the pole at $k = 0$.

Since the integrand function has two branch points $k = \mp i\alpha$, a cut along the imaginary axis from $i\alpha$ to ∞ and from $-i\alpha$ to $-\infty$, is required in order to obtain a single-valued analytic branch.

The integral along the real axis for $r > 0$ is equal to the contribution of half of the residue at the pole $k = 0$, the limit value of the integral along the following contour as $R \to \infty$ (Fig. 1), from $k = -R$ along the arc $Re^{i\theta}$ to $k = iR$, from $k = iR$ along the imaginary axis to the left of the cut to $k = i\alpha$, from $k = i\alpha$ along the imaginary axis to the right of the cut to $k = iR$, and from $k = iR$ along the arc $Re^{i\theta}$ to $k = R$ (here R denotes the radius of the circle).

Inspecting the behavior of $F(k)$ we see that as $R \to \infty$, the integral along the arcs of radius R vanishes, and only the integral along the axis to the left and right of the cut, remains.

When $r \gg a$, then the region $|k|a \ll 1$ becomes the main contributor towards the value of the integral. Therefore, using the asymptotic expression for (3.7), we obtain

$$r\varphi = \frac{3\alpha a\,(aA + 2\mu^2\alpha B)}{2\,(1 + 3\mu^2\alpha^2)} + \Psi$$

$$\Psi = \frac{\alpha a}{2}\int\limits_1^\infty \frac{[aA\,(1 - \mu^2\alpha^2 s^2) + 2\mu^2\alpha B]\exp(-\alpha r s)\,ds}{[1 - \mu^2\alpha^2 s^2 + {}^1/_2\, s^{-1}\ln(s-1)/(s+1)]^2 + {}^1/_4\,\pi^2 s^{-2}} \tag{3.11}$$

The behavior of the integrand function preceding the exponential term depends essentially on the magnitude of the parameter $\mu^2\alpha^2$.

Let $\mu^2\alpha^2 \ll 1$, then the function in question increases sharply in the region $1 < s < 1.3$ due to the behavior of $\ln(s-1)$. Then it increases slowly with increasing s reaching the maximum at the point $s \approx 1/\mu\alpha$ corresponding to the minimum value of the denominator, which is ${}^1/_4\,\pi^2\mu^2\alpha^2$. After that this function decreases in the region of the order of $1/\mu\alpha$.

If $\alpha r \gg 1$, then the small neighborhood of the point $s = 1$ makes the main contribution towards the value of the integral, consequently

$$\Psi = \frac{a}{2r}e^{-\alpha r}\int\limits_0^\infty \frac{aA\,(1 - \mu^2\alpha^2) + 2\mu^2\alpha B}{[1 - \mu^2\alpha^2 + {}^1/_2\ln(x/2\alpha r)]^2 + {}^1/_4\pi^2}\,e^{-x}dx =$$

$$= \frac{a}{2r}e^{-\alpha r}\,\frac{aA\,(1 - \mu^2\alpha^2) + 2\mu^2\alpha B}{(1 - \mu^2\alpha^2 - {}^1/_2\ln 2\alpha r)^2 + {}^1/_4\,\pi^2} \tag{3.12}$$

and the temperature distribution, in accordance with (3.11), has the form

$$r\varphi = \frac{3\alpha a\,(aA + 2\mu^2\alpha B)}{2\,(1 + 3\mu^2\alpha^2)} \tag{3.13}$$

Comparing (3.13) with the temperature field obtained from the radiant energy transfer approximation (2.14) we easily see, that the constants A and B are connected with the total energy flux by the relation

$$A + \frac{2\mu^2\alpha}{a}B = \frac{S_a}{2\sigma T_\infty^4} \tag{3.14}$$

which can be used to determine the surface temperature of the sphere.

4. Temperature field near the surface of the sphere. Eq. (3.2) gives the temperature distribution over the whole space $r > a$ with the accuracy of up to the terms of the order of αa.

On closer inspection we find that, in the region $r \sim a$ the integral term in (3.2) is of the order of αa when $\mu \leq a$, or of the order of $\alpha a \ln \alpha a$ when $\mu \gg a$. Therefore in the region $r \sim a$ we can neglect the integral term in (3.2), obtaining the following expression for the temperature field with the accuracy of up to the order of small magnitudes

$$\mu^2 \frac{d^2 r\varphi}{dr^2} = r\varphi - A\,(r - \sqrt{r^2 - a^2}) \tag{4.1}$$

When μ is sufficiently small (we show in Section 6 that the inequality $4\mu^2\alpha^2 \ll \varepsilon\alpha a$) must hold), then the solution of (4.1) becomes asymptotic already in the region where the error (which increases with increasing r) caused by neglecting the integral term of (3.2), need not be taken into account. We can therefore demand that, for $r \gg \mu$ the solution of (4.1) should not contain any terms increasing exponentially whose appearance would be inconsistent with the requirement of a smooth transition of the solution of (4.1) into the solution (3.11) valid in the region $r \gg a$.

The corresponding solution of (4.1) has the form

$$r\varphi = \left[a\varphi_a - \frac{A}{2\mu}\int_a^\infty (x - \sqrt{x^2 - a^2}) \exp\left(-\frac{x-a}{\mu}\right) dx\right] \exp\left(-\frac{r-a}{\mu}\right) +$$

$$+ \frac{A}{2\mu}\int_a^\infty (x - \sqrt{x^2 - a^2}) \exp\left(-\frac{|r-x|}{\mu}\right) dx \tag{4.2}$$

from which we have

$$r\varphi = {}^1/{}_2 a^2 A / r \quad \text{when } r \gg \mu \tag{4.3}$$

$$\left(\frac{dr\varphi}{dr}\right)_a = -\frac{a}{\mu}\varphi_a + A\left[1 + \frac{a}{\mu} - \frac{a}{\mu} e^{a/\mu} K_1\left(\frac{a}{\mu}\right)\right] \tag{4.4}$$

If $\mu \ll a$, we have

$$\left(\frac{d\varphi}{dr}\right)_a = -\frac{\varphi_a}{\mu}\left(1 + \frac{\mu}{a}\right) + \frac{A}{\mu}\left(1 + \frac{\mu}{a} - \sqrt{\frac{\pi\mu}{2a}}\right) \tag{4.5}$$

In the limit as $\mu \to 0$, the temperature distribution undergoes a jump. When μ has a finite value $(\mu \ll a)$, the temperature distribution varies sharply within the region $r - a \sim \mu$. At the distances $r - a \gg \mu$, the temperature distribution is given by (4.3).

If $\mu \gg a$, we have

$$\left(\frac{d\varphi}{dr}\right)_a - \frac{\varphi_a}{a}\left(1 + \frac{a}{\mu}\right) + \frac{Aa}{2\mu^2}\left(\ln\frac{a}{2\mu} - 0.427\right) \tag{4.6}$$

When $r \sim a$, the temperature field differs little from that obtained in the absence of radiation and in this case we can assume that $B = \varphi_a$.

5. Radiation equilibrium. Equation defining the temperature around the radiating sphere for a medium with the vanishing small molecular heat transfer coefficient $4\mu^2\alpha^2 \ll \varepsilon\alpha a$, can be obtained either from the general equation (1.8), or from (3.2) by putting $\varkappa = 0$

$$r\varphi = A\,(r - \sqrt{r^2 - a^2})\, e^{-\alpha r} + \frac{\alpha}{2}\int_a^\infty \rho\varphi\, d\rho \int_{|r-\rho|}^{r+\rho} e^{-\alpha u}\frac{du}{u} \tag{5.1}$$

Unlike (3.2), Eq. (5.1) is valid for any value of φ_a, provided that $\varkappa = 0$.

Solution (3.11) of (3.2) obtained earlier for the region $r \ll a$, becomes

$$r\varphi = \frac{\alpha a^2 A}{2}\left(3 + \int_1^\infty \frac{\exp(-\alpha r s)\, ds}{[1 + {}^1/{}_2 s^{-1} \ln(s-1)/(s+1)]^2 + {}^1/{}_4 \pi^2 s^{-2}}\right) \tag{5.2}$$

as $\mu \to 0$.

At the distances from the sphere which are large compared with the path length of the photons $(\alpha r \gg 1)$ we have, in accordance with (3.12),

$$r\varphi = {}^1/_2\ \alpha a^2 A\ \{3 + (\alpha r)^{-1}[({}^1/_2 \ln 2\alpha r - 1)^2 + {}^1/_4\ \pi^2]^{-1} \exp(-\alpha r)\} \qquad (5.3)$$

In the region $a \ll r \ll 1/\alpha$, Formula (5.2) simplifies since the function preceding the exponential term can be replaced by unity

$$r\varphi = {}^1/_2\ \alpha a^2 A\ [3 + (\alpha r)^{-1}] = {}^1/_2\ a^2 A / r \qquad (5.4)$$

This agrees with the temperature distribution (4.3) near the surface of the sphere, provided that $r \gg \mu$.

Using (5.4) we can easily obtain the following estimate:

$$\frac{\alpha}{2}\int_a^\infty \rho\varphi d\rho \int_{|r-\rho|}^{r+\rho} e^{-\alpha u}\frac{du}{u} < \frac{\alpha}{2}\int_a^\infty \frac{a^2 A}{2\rho} \ln\frac{r+\rho}{|r-\rho|} d\rho < \frac{\pi^2}{8}\alpha a^2 A$$

Thus, when $\alpha r \ll 1$, then the solution of (5.1) has the form

$$r\varphi = A\,(r - \sqrt{r^2 - a^2}) \qquad (5.5)$$

up to the surface of the sphere $r = a$ and with the accuracy of up to αa.

Both results, (5.5) and (5.4) are in complete accord. We can also construct the interpolation equation

$$r\varphi = A\,[(r - \sqrt{r^2 - a^2}) \exp(-\alpha r) + {}^3/_2 \alpha a^2] \qquad (5.6)$$

valid for the regions $a < r \ll 1/\alpha$, $r \gg 1/\alpha$.

The limit temperature of the medium as $r \to a$, in accordance with (5.5), and with the accuracy of up to the order of αa, is defined by

$$\lim_{r\to a} r\varphi = aA \neq a\varphi_a \qquad (5.7)$$

and this confirms the temperature jump occurring at the surface of the sphere, since we have by (3.2) and (3.14) and with the accuracy of up to the order of αa

$$A = \frac{1}{2}\varepsilon\left[\varphi_a - \alpha\int_a^\infty \varphi \exp(-\alpha\rho)\, d\rho\right] \approx \frac{1}{2}\varepsilon\varphi_a = \frac{S_a}{2\sigma T_\infty^4} \qquad (5.8)$$

Indeed, (5.5) shows that the principal contribution towards the integral appearing in (5.8) is made by the region $\rho \sim a$.

Thus the temperature undergoes a jump at the surface of the sphere, and its value is given by

$$T_a^4 - T_{a+0}^4 = (1 - {}^1/_2\varepsilon)(T_a^4 - T_\infty^4) \qquad (5.9)$$

6. Temperature field and the radiant energy flux when $\mu^2\alpha^2 \ll 1$, $\mu \gg a$. If $\alpha r \ll 1$ and $r \gg \mu$, then the condition $\mu^2\alpha^2 s^2 \ll 1$ holds in the region, in which the integrand function in (3.11) is different from zero and we have

$$\Psi = {}^1/_2\ (a/r)(aA + 2\mu^2\alpha\varphi_a) \qquad (6.1)$$

consequently, for the whole region $r \gg \mu$ the temperature field is given by the following interpolation formula:

$$r\varphi = {}^3/_2\ \alpha a\ (aA + 2\mu^2\alpha\varphi_a)\ [1 + {}^1/_3\ (\alpha r)^{-1} \exp(-\alpha r)] \qquad (6.2)$$

If $r \sim \mu$, then the small neighborhood of the maximum point of the integrand function makes the essential contribution towards Ψ, and we have, with the accuracy of up to the terms of the order of $\mu\alpha$,

$$\Psi = \frac{\alpha a}{2}\exp\left(-\frac{r}{\mu}\right)\int_1^\infty \frac{aA\,(1 - \mu^2\alpha^2 s^2) + 2\mu^2\alpha\varphi_a}{(1 - \mu^2\alpha^2 s^2)^2 + {}^1/_4\pi^2\mu^2\alpha^2}\, ds =$$

$$= \frac{a}{4\mu} \exp\left(-\frac{r}{\mu}\right) \int\limits_{-\infty}^{\infty} \frac{aA(1-t^2) + 2\mu^2\alpha\varphi_a}{(1-t^2)^2 + {}^1/_4\pi^2\mu^2\alpha^2} dt = a\varphi_a \exp\left(-\frac{r}{\mu}\right) \qquad (6.3)$$

Comparing it with the solution (4.2) for $r \sim a$ we see, that (6.3) is also valid in this region.

From (3.2), (6.2) and (6.3) it follows that $A = {}^1/_2\ \varepsilon\varphi_a$ to within the terms of the order of $\alpha a \ln \alpha a$.

By (3.14) the total energy flux density is

$$S_0 = \varepsilon\sigma \frac{a^2}{r^2}(T_a^4 - T_\infty^4) + \frac{\varkappa a}{r^2}(T_a - T_\infty) \qquad (6.4)$$

We can determine the radiant energy flux using the expression for the total energy flux incorporating both, radiant and molecular energy transfer terms. For this purpose we must compute the temperature gradient first, and then use (6.4) to obtain

$$S = S_0 (1 - 2\mu^2\alpha/r) \qquad (6.5)$$

for $\mu \ll r \ll 1/\alpha$.

We see that $S \approx S_0$, i.e. the total energy flux, as in the region $\alpha r \gg 1$, is basically defined by the radiation. When $r \leqslant \mu$, then analogous computations with the aid of (6.3) yield the following result:

$$S = S_0 \left[1 - \frac{1 + r/\mu}{1 + \varepsilon a / 4\mu^2\alpha} \exp\left(-\frac{r}{\mu}\right)\right] \qquad (6.6)$$

which shows that the contributions of the radiant and molecular flux are independent of distance over the whole region $r \ll \mu$. We see from (6.6) and (6.5) that the contribution of the molecular energy transfer flux to the total flux decreases, in the region $r \sim \mu$ with increasing r. When $r \gg \mu$, then the total energy flux is defined by the radiation only. The expression $4\mu^2\alpha/\varepsilon a$ is a parameter defining the relative importance of the radiant and the molecular transfer in the region $r \ll \mu$. When this parameter is considerably less than unity, the molecular energy transfer can be neglected everywhere. If $4\mu^2\alpha/\varepsilon a \gg 1$, then the energy transfer is purely molecular in the region $r \ll \mu$ the mechanism however becomes radiant when $r \gg \mu$. Variation in the radiant energy flux at the distances $r \sim \mu \ll 1/\alpha$ is governed by the luminosity of the sphere.

7. Temperature and the radiant energy flux when $\mu^2\alpha^2 \gg 1$. At large distances from the sphere $(\alpha r \gg 1)$ the proportion of the radiant and molecular energy fluxes is defined, as seen from (2.13), by the value of the coefficient $3\mu^2\alpha^2$.

If $\mu^2\alpha^2 \gg 1$, then, by (3.11), the integral becomes

$$\Psi = (a\varphi_a/\mu^2\alpha^2)E_4(\alpha r) \qquad (7.1)$$

with the accuracy of up to order of αa and for any value of r except those within the region $\ln \alpha r \gg \mu^2\alpha^2$.

Temperature distribution over the whole space is given by

$$r\varphi = a\varphi_a [1 - {}^1/_3\ (\mu\alpha)^{-2} + (\mu\alpha)^{-2}E_4(\alpha r)] \qquad (7.2)$$

according to which we have

$$r\varphi = a\varphi_a (1 - \alpha r/6\mu^2\alpha^2) \quad \text{when } \alpha r \ll 1 \qquad (7.3)$$

$$r\varphi = a\varphi_a [1 - {}^1/_3\ (\mu\alpha)^{-2} + \mu^{-2}\alpha^{-3}r^{-1} \exp(-\alpha r)] \quad \text{when } \alpha r \gg 1$$

Molecular energy flux density is

$$S_\varkappa = -\varkappa \frac{dT}{dr} = \frac{\varkappa a}{r^2}(T_a - T_\infty)\left[1 - \frac{1}{3\mu^2\alpha^2} + \frac{E_4(\alpha r) + \alpha r E_3(\alpha r)}{\mu^2\alpha^2}\right] \qquad (7.4)$$

At the large distances where the diffusion approximation is valid for $\mu^2\alpha^2 \gg 1$ we can use the expression for $S_\varkappa$ at $\alpha r \gg 1$ to obtain the total energy flux density

$$S_0 = S_\varkappa\left(1 + \frac{1}{3\mu^2\alpha^2}\right) = \frac{\varkappa a}{r^2}(T_a - T_\infty) \qquad (7.5)$$

This implies that the radiant energy flux density over the whole space is

$$S = \frac{16\sigma a T_\infty^3}{\alpha r^2}\left[\frac{1}{3} - E_4(\alpha r) - \alpha r E_3(\alpha r)\right](T_a - T_\infty) \qquad (7.6)$$

We see here that in the region $\alpha r \ll 1$ the influence of the molecular mode of the energy transfer is dominant and the radiant energy flux is practically absent. When $\alpha r \gg 1$, the flux tends exponentially to the limit defined by the radiant tranfer approximation, and is a small quantity of the order of $(\mu\alpha)^{-2}S_\varkappa$.

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INFINITE ELASTIC LAYER AND HALF-SPACE UNDER THE ACTION OF A RING-SHAPED DIE

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G. M. VALOV
(Kostroma)
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The problem of pressure due to an axisymmetric ring-shaped die on an elastic half-space and layer was examined in [1 and 2]. In these papers the boundary value problem of the theory of elasticity is reduced to a linear integral equation of the second kind with a kernel given by a set of infinite measure. In [3] the problem of pressure due to a ring-shaped die on an elastic layer is reduced to a Fredholm integral equation of the second kind by means of approximate substitution of the kernel of the integral equation of the first kind. Normal stresses under the die are expressed through the derivative of the solution of this equation. In papers [4 and 5] the problem of pressure due to an axisymmetric ring-shaped die on a half-space was solved by approximate methods.

In this paper the axisymmetric problem of pressure due to a ring-shaped die on an infinite elastic layer and half-space is solved and also the problem of torsion of the elastic layer and half-space under the influence of a coupled rigid die. In addition to the die, the half-space and layer are under the influence of a steady-state temperature field. The solution of boundary value problems are presented in the form of integrals which